Microeconomic Theory
Graduate Lecture Notes

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Contents

Remarks

- Notes prepared for the SSE shared course, Oct.–Nov. 1996
- Revised for Haifa University
- For a Syllabus see a separate handout
- Text: Mas-Colell, A., Whinston, M., and J. Green. 1995. *Microeconomic Theory*. Oxford, Oxford University Press.
- Lecture is 2×45 minutes (Stockholm)
- Lecture is 3×45 minutes nonstop (Haifa)

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Lecture 1

Mathematical Background: Correspondences, Continuity, and Fixed-Point Theorems

1.1 Notation

- Let $x, y \in \mathbb{R}^N$ (i.e., $x = (x_1, \ldots, x_N)$ etc.
  - $x \geq y$ means $x_i \geq y_i \quad \forall i = 1, \ldots, N$
  - $x \gg y$ means $x_i > y_i \quad \forall i = 1, \ldots, N$
  - $x \neq y$ means that $\exists i \in \{1, \ldots, N\}$ with $x_i \neq y_i$.
  - $\exists$ there exists
  - $\forall$ for all

1.2 Some topology

- Let X and Y be subsets in $\mathbb{R}^N$ (i.e., $X \subseteq \mathbb{R}^N$ and $Y \subseteq \mathbb{R}^N$)
  - X is a *subset* of Y, $X \subseteq Y$, if $x \in X \Rightarrow x \in Y$
  - X is a *proper subset* of Y, $X \subset Y$, if $x \in X \Rightarrow x \in Y$ and $\exists y \in Y$ with $y \notin X$.
- An *open ball* (of size ϵ) around a vector $x \in \mathbb{R}^N$ is the set

$$B_\epsilon(x) \overset{\text{def}}{=} \left\{ y \in \mathbb{R}^N : \|y - x\| < \epsilon \right\}$$

  where $\|x\|$ is the "norm of x,"
  e.g., in $\mathbb{R}^2$, $\|y - x\| = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}$.
- A set $A \subseteq \mathbb{R}^N$ is called *open* if

$$\forall x \in A, \ \exists \epsilon > 0 \text{ such that } B_\epsilon(x) \subset A$$

- A set $A \subset \mathbb{R}^N$ is *closed* if for every converging sequence $\{x^m\}_{m=1}^\infty$ where $x^m \in A, \forall m$ and $x^m \to x$, then $x \in A$.
- A set $A \subset \mathbb{R}^N$ is *bounded* if $\exists r \in \mathbb{R}$ such that $\|x\| < r, \forall x \in A$, e.g., in $\mathbb{R}^2$ if $\sqrt{(x_1)^2 + (x^2)^2} < r$
- A set $A \subset \mathbb{R}^N$ is *compact* if it is closed & bounded
- A set $A \subset \mathbb{R}^N$ is *convex* if

$$\forall x, y \in A, \quad \lambda x + (1 - \lambda) y \in A, \quad \forall 0 < \lambda < 1$$

1.3 Correspondences (not needed if I use Brouwer's Theorem)

A *correspondence* is a set-valued function (i.e, the domain could be a set not just a point)

DEFINITION 1.1 *Let $A \subseteq \mathbb{R}^N$. A **correspondence** is a rule $f : A \to \mathbb{R}^K$ which assigns a set $f(x) \subseteq \mathbb{R}^K, \forall x \in A$.*

DEFINITION 1.2 *A **graph** of the correspondence f is the set $G \subseteq \mathbb{R}^{N+K}$,*
$G \stackrel{def}{=} \{(x, y) \in A \times Y | y \in f(x)\}$

DEFINITION 1.3 *Let $A \subseteq \mathbb{R}^N$ and a closed set $Y \subseteq \mathbb{R}^K$. The correspondence $f : A \to Y$ has a **closed graph** if for any two sequences $\{x^m\}_{m=1}^{\infty}$ and $\{y^m\}_{m=1}^{\infty}$ with*

$$x^m \to x \in A, \quad \textit{and} \quad y^m \in f(x^m)$$

Then

$$\textit{if } y^m \in f(x^m) \; \forall m \quad \textit{then } y \in f(x)$$

Note: Show figures on page 950
Attention: Closed graph does not necessarily imply a bounded graph!

DEFINITION 1.4 *A correspondence $f : A \to Y$ is **upper semicontinuous (usc)** if*

1. *its graph is closed*

2. *every compact set $B \subseteq A$ has a bounded image set; that is the set*

$$f(B) = \{(y \in Y | y \in f(x) \textit{ for some } x \in B\} \quad \textit{is bounded}$$

Remark: In $\mathbb{R}^N$ upper semicontinuous is the same as upper hemicontinuous (as defined in the Text). The following proposition is brought without a proof.

Proposition 1.1 *If $f : A \to Y$ is singled valued (i.e., a function) then f is upper semicontinuous if and only if it is continuous.*

1.4 Fixed-Point Theorems

DEFINITION 1.5 *Let $A \subseteq \mathbb{R}^N$, and let the correspondence $f : A \to A$ (maps a set to itself). Then, the vector x^* is a **fixed point** of f if $x^* \in f(x^*)$.*

Now, Kakutani's Fixed-Point Theorem:

Theorem 1.1 *Let $A \subseteq \mathbb{R}^N$ satisfying*

1. *$A \neq \emptyset$*

2. *A compact*

3. *A convex*

and let $f : A \to A$ be an USC correspondence satisfying $\forall x \in A$, the set $f(x) \neq \emptyset$ and convex. Then, f has a fixed point in A.

Note: show figures on page 953

Now, Brouwer's Fixed-Point Theorem:

Theorem 1.2 *Let $A \subseteq \mathbb{R}^N$ satisfy*

1. $A \neq \emptyset$
2. *A compact*
3. *A convex*

Let the function $f : A \mapsto A$ be continuous. Then,

$$\exists x \in A, \quad \textit{such that} \quad x = f(x)$$

Why convexity and compactness are needed? Show counter examples:

Convexity: Let $A \stackrel{\text{def}}{=} [0,1] \cup [2,3]$

Compactness: Let $f : [0,1] \mapsto \mathbb{R}$ so $f(x) \stackrel{\text{def}}{=} 1 - \sqrt{x}$.

1.5 Separating Hyperplanes

Definition 1.6 *Given $p \in \mathbb{R}^N$, with $p \neq 0$ (meaning that $\exists i \in \{1, \ldots, N\}$ with $p_i \neq 0$). Let $c \in \mathbb{R}$. The **hyperplane** generated by p and c is the set*

$$H_{p,c} \left\{ x \in \mathbb{R}^N | px = c \right\}$$

*The spaces above or below the hyperplane are called **half spaces**.*

Without proving, the separating hyperplane theorem states as follows:

Theorem 1.3 *Let $B \in \mathbb{R}^N$ be convex and closed. Let $x \notin B$. Then,*

$$\exists p \in \mathbb{R}^N, p \neq 0, \textit{ and } c \in \mathbb{R} \quad \textit{s.t. } px > c \textit{ and } py < c \, \forall y \in B$$

Hence, Let $A, B \subset \mathbb{R}^N$ convex and satisfy $A \cap B = \emptyset$. Then

$$\exists p \in \mathbb{R}^N, p \neq 0, \textit{ and } c \in \mathbb{R} \quad \textit{s.t. } px \geq c \, \forall x \in A \quad \textit{and} \quad px \leq c \, \forall x \in B$$

Lecture 2
General Competitive Equilibrium

2.1 Major Issues

- Agents: consumers and producers
- Price taking behavior for all agents
- Does not necessarily imply very large number of agents
- Commonly called Arrow-Debreu economy to imply some welfare property
- Free disposal
- Purpose of this analysis:
  1. Find the conditions under which equilibrium exists and is unique
  2. Find welfare property of a competitive equilibrium

  i.e., dealing with positive and normative aspects

2.2 Existence of a Competitive Equilibrium

- L goods
- J firms indexed by $j = 1, \ldots, J$
- Each firm j is a production set $Y_j \subseteq \mathbb{R}^L$
- I consumers indexed by $i = 1, 2, \ldots, I$
- Each consumer i is characterized by
  1. consumption set $X_i \subseteq \mathbb{R}^L$
  2. initial endowment (Manna) $\omega_i \in X_i$
  3. binary preference relation $\succeq_i$ over X_i
  4. θ_{ij}, $0 \leq \theta_{ij} \leq 1$ which is the ownership share of consumer i in firm j. (*Note:* $\forall j$, $\sum_{i=1}^{I} \theta_{ij} = 1$)

Definition 2.1 *A **private-ownership economy** is the set (of sequences of sets)*

$$\mathcal{E} = \left\{ \{(X_i, \succeq_i), \omega_i, \theta_{i1}, \ldots, \theta_{iJ}, \}_{i=1}^{I}, \{Y_j\}_{j=1}^{J} \right\}$$

Definition 2.2 *An **allocation** for this economy is a collection of consumption and production vectors*

$$(x_1, \ldots, x_I, y_1, \ldots, y_J) \in X_1 \times \cdots \times X_I \times Y_1 \times \cdots \times Y_J$$

Definition 2.3 *An allocation* $(x_1^c, \ldots, x_I^c, y_1^c, \ldots, y_J^c)$ *and a price vector* $p^c \stackrel{def}{=} (p_1^c, \ldots, p_L^c)$ *constitute a **competitive equilibrium** if*

Profit Maximization: *For every firm* j*,* $y_i^c \in Y_j$ *satisfies*

$$p^c y_j^c \geq p^c y_j, \quad \forall y_j \in Y_j$$

Utility Maximization: *For every consumer* i*,* $x_i^c \in X_i$ *satisfies*

$$x_i^c \succeq_i x_i, \quad \forall x_i \in B_i(p^c) \stackrel{def}{=} \left\{ x_i \in X_i | p^c x_i \leq p^c \omega_i + \sum_{j=1}^{J} \theta_{ij} p^c y_j^c \right\}$$

Feasibility:

$$\sum_{i=1}^{I} x_i^c \leq \sum_{i=1}^{I} \omega_j^c + \sum_{j=1}^{J} y_j^c$$

Assumption 2.1 *The economy* $\mathcal{E}$ *is a pure-exchange economy with no production, and free disposal.*

Definition 2.4 *Let* $p \in \mathbb{R}_+^L$*. The **excess demand** of consumer* i*,* $z_i(p) = (z_{i1}, \ldots, z_{iL})$ *is*

$$z_i(p) \stackrel{def}{=} x_i(p, p\omega_i) - \omega_i$$

*The **economy's aggregate excess demand** function is*

$$z(p) \stackrel{def}{=} \sum_{i=1}^{I} z_i(p)$$

Hence, in a pure exchange economy, competitive equilibrium means a price, p^c, satisfying $z(p) \leq 0$.

Definition 2.5 *The **unit simplex** is the set of prices,* Δ*, satisfying*

$$\Delta \stackrel{def}{=} \left\{ p \in \mathbb{R}_+^L : \sum_{\ell=1}^{L} p_\ell = 1 \right\}$$

The set of all strictly positive prices in the simplex is called

$$Int\Delta \stackrel{def}{=} \{p \in \Delta : p_\ell > 0 \; \forall \ell\}$$

2.2.1 Existence using Kakutani's Theorem (ignore)

Assumption 2.2

1. *$z(p)$ is continuous*
2. *$z(p)$ is homogeneous of degree zero*
3. *$pz(p) = 0$ for all p [Walras' Law]*
4. *$\exists s > 0$ such that $z(p) > (-s, -s, \ldots, -s)$ for all p [lower bound]*

5. *Let* $p \in \mathbb{R}^L_+$, $p \neq 0$ *but* $p_\ell = 0$ *for some* ℓ. *Then,*

$$\textit{if } p^m \to p, \quad \textit{then } \max_\ell \{z_1(p^m), \ldots, z_L(p^m)\} \to \infty$$

Our main proposition:

Theorem 2.1 *Let* $z(p)$ *be defined for all* $p \in \mathbb{R}^L_{++}$ *and satisfy Assumption 2.2. Then,* $\exists p^c \gg 0$ *such that* $z(p^c) = 0$

Proof. It is sufficient to restrict a solution to $p^c \in \text{Int}\Delta$ (by homogeneity). Define the correspondence $f : \Delta \mapsto \Delta$ by

$$f(p) \stackrel{\text{def}}{=} \begin{cases} \{q \in \Delta | qz(p) \geq q'z(p), \ \forall q' \in \Delta & \text{if } p \in \text{Int}\Delta \\ \{q \in \Delta | pq = 0\} = \{q \in \Delta | q_\ell = 0 \text{ if } p_\ell > 0\} & \text{if } p \notin \text{Int}\Delta \end{cases}$$

Note: To have a compact domain, we need to define f on the entire Δ. The separation to interior and boundary of the simplex is made because $z(p)$ is defined only on Int Δ.

Lemma 2.1 *For all* $p \in \Delta$, $f(p)$ *is a convex set*

Proof. Let $p \in \Delta$ be given. $q' \in f(p)$, $q'' \in f(p)$.
If $p \in \text{Int}\Delta$,

$$z(p)[\lambda q' + (1-\lambda)q''] = \lambda z(p)q' + (1-\lambda)z(p)q'' \geq \lambda z(p)q + (1-\lambda)z(p)q = z(p)q, \ \forall q \in \Delta$$

If $p \neq \text{Int}\Delta$, If $p_\ell > 0$, then $q'_\ell = q''_\ell = 0$, hence $\lambda q' + (1-\lambda)q'' \in f(p)$. ■

Lemma 2.2 $f(p)$ *is upper hemicontinuous*

Proof. Let $p^m \to p$. Let $q^m \in f(p^m)$ with $q^m \to q$. We need to show $q \in f(p)$.

We ignore the case $p \notin$Int Δ (for this part we need to use the last two conditions in Assumption 2.2). Hence, for m sufficiently large, $p^m \gg 0$.
Since $z(p)$ is continuous,

$$q^m z(p^m) \geq q'z(p), \forall q' \in \Delta \Longrightarrow qz(p^m) \geq q'z(p), \forall q' \in \Delta$$

■

Altogether, since
(i) $\Delta \neq \emptyset$, convex, and compact set
(ii) $f : \Delta \mapsto \Delta$ is convexed-valued & upper hemicontinuous
Kakutani's fixed-point theorem applies, so

$$\exists p^c \in \Delta, \quad \text{such that} \quad p^c \in f(p^c).$$

■

2.2.2 Existence using Brouwer's Theorem

Assumption 2.3 *1. $z(p)$ is continuous*

2. $z(p)$ is homogeneous of degree zero

3. $pz(p) = 0$ for all p [Walras' Law]

Our main proposition:

Theorem 2.2 *Let $z(p)$ be defined for all $p \in \mathbb{R}_+^L$, $p \neq 0$, and satisfy Assumption 2.3. Then, $\exists p^c \in \mathbb{R}_+^L$ such that $z(p^c) = 0$*

Proof. Define the function $f = (f_1, \ldots, f_L) : \Delta \mapsto \Delta$:

$$f_\ell(p) \stackrel{\text{def}}{=} \frac{p_\ell + \max\left(z_\ell(p), 0\right)}{1 + \sum_{\ell=1}^L \max(z_\ell(p), 0)}$$

Note that

1. f is continuous (since max is continuous
2. f maps into Δ since $\sum_\ell f_\ell = 1$
3. $f_\ell(p) \nearrow$ when $z_\ell(p) > 0$ (i.e. the function f increases with the price upon excess demand for good ℓ).

Hence,

$$\exists p^c \in \Delta \quad \text{such that} \quad p^c = f(p^c) \quad \text{hence} \quad p_\ell^c = \frac{p_\ell^c + \max\left(z_\ell(p^c), 0\right)}{1 + \sum_{\ell=1}^L \max(z_\ell(p^c), 0)} \quad \ell = 1, \ldots, L$$

Cross multiplying and multiplying both sides by $z(p^c)$ yields

$$z_\ell(p^c) p_\ell^c \left[\sum_{\ell=1}^L \max(z_\ell(p^c), 0)\right] = z_\ell(p^c) \max(z_\ell(p^c), 0)$$

Summing over all ℓ

$$\left[\sum_{\ell=1}^L \max(z_\ell(p^c), 0)\right] \sum_{\ell=1}^L p_\ell^c z_\ell(p^c) = \sum_{\ell=1}^L z_\ell(p^c) \max(z_\ell(p^c), 0)$$

LHS is zero by Walras' Law. Hence,

$$\sum_{\ell=1}^L z_\ell(p^c) \max(z_\ell(p^c), 0) = 0$$

Each term is either 0 or $(z_\ell(p^c))^2 > 0$. Hence, $z_\ell(p^c) = 0$. ■

2.2.3 Examples

Taken from textbook p. 521:

$$u^1 = x_1^1 - \frac{(x_2^1)^{-8}}{8} \quad u^2 = -\frac{(x_1^2)^{-8}}{8} + x_2^2 \quad \omega^1 = (2, r) \quad \omega^2 = (r, 2), \quad r = 2^{\frac{8}{9}} - 2^{\frac{1}{9}}$$

Solve for competitive equilibria!

$$MRS^1 = \frac{MU_1^1}{MU_2^1} = (x_2^1)^9 = \frac{p_1}{p_2} \Longrightarrow x_2^1 = \sqrt[9]{\frac{p_1}{p_2}}$$

Budget constraint:

$$p_1 x_1^1 + p_2 x_2^1 = p_1 x_1^1 + p_2 \sqrt[9]{\frac{p_1}{p_2}} = 2p_1 + p_2 r$$

Hence, his demand function for good 1 is:

$$x_1^1(p_1, p_2, \omega_1) = \frac{2p_1 - (p_1)^{1/9} p_2^{8/9} + p_2 r}{p_1}$$

For consumer 2:

$$MRS^2 = \frac{MU_1^2}{MU_2^2} = \frac{1}{(x_1^2)^9} = \frac{p_1}{p_2} \Longrightarrow x_1^2(p_1, p_2) = \sqrt[9]{\frac{p_2}{p_1}}$$

Equilibrium in the market for good 1:
$x_1^1 + x_1^2 = 2 + r$, or $z_1(p_1, p_2) \stackrel{\text{def}}{=} x_1^1 + x_1^2 - (2 + r) = 0$. Hence,

$$z_1(p_1, p_2) = \frac{2p_1 - (p_1)^{1/9} p_2^{8/9} + p_2 r}{p_1} + \sqrt[9]{\frac{p_2}{p_1}} - (2 + r) = 0$$

Or,

$$z_1(p_1, p_2) = \left(\frac{p_2}{p_1}\right)^{1/9} - \left(\frac{p_2}{p_1}\right)^{8/9}) + \frac{p_2 r}{p_1} - r$$

which has 3 solutions: $p_1 \in \{1/2, 1, 2\}$ (see Figure 2.1).

2.3 Uniqueness of Competitive Equilibrium

- Uniqueness is essential for making the prediction of equilibrium values meaningful and credible.
- For the case of pure-exchange economy, there are two sufficient condition that are generally sufficient for uniqueness:
    1. assuming that the aggregate economy excess demand functions $z(p)$ satisfy the Weak Axion of Revealed Preference, i.e.,

$$\forall p, p' \in \mathbb{R}_+^L, \quad \text{if } z(p) \neq z(p') \ \& \ pz(p') \leq 0 \Longrightarrow p'z(p) > 0 \ \text{ not affordable}$$

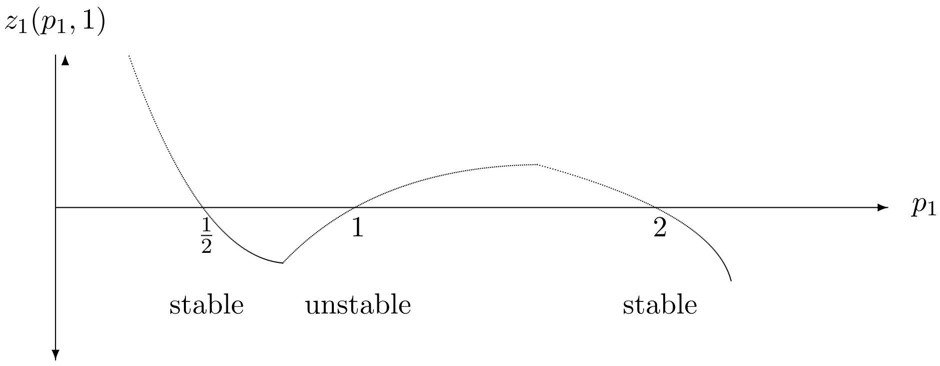

Figure 2.1: Multiple equilibria and stability

2. assuming Gross Substitution (i.e., an increase in price of one good increases demand for all other goods).

Remark: Either condition is very strong

Definition 2.6 *The function $z(p)$ has the* ***gross substitution*** *property (GS) if whenever p and p' satisfy $p_\ell > p'_\ell$ for some ℓ, and $p_k = p'_k \forall k \neq \ell$, then $z_k(p) > z_k(p')\ \forall k \neq \ell$.*

Proposition 2.1 *Let $z(p)$ satisfy the gross substitution property. Then $z(p) = 0$ has at most one solution.*

Proof. By a way of contradiction suppose that $\exists p, p' \in \mathbb{R}^L_+,\ p \neq kp'$ for some $k > 0$ such that $z(p) = z(p') = 0$.

By homogeneity of $z(p)$, we can assume that $p \geq p'$ and $p_\ell > p'_\ell$ for some ℓ.
By (GS), $z_\ell(p) < z_\ell(p') = 0$ for some ℓ. A contradiction. ■

2.4 Dynamic Stability of Equilibrium

- Uniqueness is related to stability, both mathematically and for the purpose of meaningful prediction of the model
- In the case of multiple equilibria, stability can rule which one to adopt as a predictor.

The following assumption was proposed in Samuelson, 1947, *Foundations of Economic Analysis*:

Assumption 2.4 *The price-tâtonnement process takes the form of*

$$\frac{dp_\ell}{dt} \stackrel{def}{=} c_\ell z_\ell(p) \quad \forall \ell = 1, \ldots, L$$

Proposition 2.2 *Let p^c satisfy $z(p^c) = 0$, and let $z(p)$ satisfy WARP. Every solution to the dynamic process given in Assumption 2.4 converges to p^c.*

Proof. Take the Euclidean distance between $p(t)$ and p^c, i.e,

$$f(p) \stackrel{\text{def}}{=} \sum_{\ell=1}^{L} \frac{1}{2c_\ell}(p_\ell - p_\ell^c)^2$$

Then,

$$\frac{df(p)}{dt} = \sum_{\ell=1}^{L} \frac{p_\ell(t) - p_\ell^c}{c_\ell} \frac{dp_\ell(t)}{dt} =$$

$$= \sum_{\ell=1}^{L} [p_\ell(t) - p_\ell^c] z_\ell(p(t)) = 0 - p^c z(p(t)) \leq 0,$$

where the last step follows by WARP, since $p^c z(p^c) = 0$, so $z(p(t))$ is not affordable. ■

2.5 Pareto-Optimal Allocations

- The normative approach asking: which allocation are 'efficient'
- Problem: efficient for whom (entering social philosophy)
- The Pareto criterion is weak in the sense that it does not involve value judgement
- i.e., distribution questions are set a side. Only 'waste' is considered.

DEFINITION 2.7 *An allocation $(x, y) = (x_1, \ldots, x_I, y_1, \ldots, y_J)$ said to be **feasible** if*

$$\sum_{i=1}^{I} x_{i\ell} \leq \sum_{i=1}^{I} w_{i\ell} + \sum_{j=1}^{J} y_{j\ell}$$

DEFINITION 2.8 *An allocation (x, y) is said to be **Pareto Optimal (Efficient)** if there does not an allocation (x', y') which Pareto dominates it, i.e., that there does not exist a feasible allocation (x', y') such that*

$$\forall i, x_i' \succeq x_i, \quad \text{and} \quad \exists i \text{ for which } x_i' \succ x_i$$

2.6 The Fundamental Theorems of Welfare Economics

The 'building block' of capitalism and decentralization doctrines.
Assumptions:

1. no externalities
2. no joint production
3. local nonsatiation (or monotonicity)
4. competitive behavior of all agents

The First Welfare Theorem of Classical Economics is:

Theorem 2.3 *If preferences are locally-nonsatiated and if* (x^c, y^c, p^c) *is a competitive equilibrium, then the allocation* (x^c, y^c) *is Pareto optimal.*

Proof. By a way of contradiction, suppose not. Then, there exists a feasible allocation (x, y) with

$$\forall i, x_i \succeq x_i^c, \quad \text{and} \quad \exists i \text{ for which } x_i \succ x_i^c$$

Define

$$\tilde{W}_i \stackrel{\text{def}}{=} p^c w_i + \sum_j \theta_{ij} p_j^c y_j$$

Now, since (x^c, y^c, p^c) is a price equilibrium

$$\text{if } x_i \succ_i x_i^c \quad \text{then} \quad p^c x_i > \tilde{W}_i$$

Lemma 2.3

$$\textit{if } x_i \succeq_i x_i^c \quad \textit{then} \quad p^c x_i \geq \tilde{W}_i.$$

Proof. By a way of contradiction suppose not. Then,

$$\exists x_i \in X_i \text{ such that } x_i \succeq_i x_i^c \text{ but } p^c x_i < \tilde{W}_i$$

By local nonsatiation,

$$\exists \epsilon > 0 \text{ and } x_i' \in B_\epsilon(x_i) \text{ s.t. } x_i' \succ_i x_i$$

By transitivity of $\succeq$

$$x_i' \succ_i x_i \succeq_i x_i^c \Longrightarrow x_i' \succ_i x_i^c$$

A contradiction. ■

Using these two properties, summing over consumers

$$\sum_{i=1}^{I} p^c x_i > \sum_{i=1}^{I} p^c w_i + \sum_{j=1}^{J} p^c y_j^*$$

However,

$$\sum_{i=1}^{I} x_i = \sum_{i=1}^{I} w_i + \sum_{j=1}^{J} y_j \Longrightarrow \sum_{i=1}^{I} p^c x_i = \sum_{i=1}^{I} p^c w_i + \sum_{j=1}^{J} p^c y_j^*$$

A contradiction (to feasibility). ■

The Second Welfare Theorem:

Theorem 2.4 *Let*

1. $\succeq_i$ *a be convex and nonsatiated preference relation*

2. Y_j *be convex production sets.*

Let (x^*, y^*) *be a Pareto optimal allocation. Then, there exist an endowments* $\omega_1, \ldots, \omega_I$ *for which there exists a price vector* $p^c \neq 0$ *where* $\left(\{x_i^*\}_{i=1}^{I}, \{y_j\}_{j=1}^{J}, \{p^c\}_{\ell=1}^{L}\right)$ *is a competitive equilibrium.*

Proof. Define,

$$V_i \stackrel{\text{def}}{=} \{x_i \in X_i | x_i \succeq_i x_i^*\} \subset \mathbb{R}^L$$

$$V \stackrel{\text{def}}{=} \sum_{i=1}^{I} V_i = \left\{ \sum_{i=1}^{I} x_i | x_1 \in V_1, \ldots, x_I \in V_I \right\}$$

$$Y \stackrel{\text{def}}{=} \sum_{j=1}^{J} Y_j = \left\{ \sum_{i=1}^{I} y_i | y_1 \in Y_1, \ldots, y_J \in Y_J \right\}$$

Lemma 2.4 *The sets $Y + \sum_i \omega_i$ and V are convex.*

Proof. The sum of convex sets is convex. It is left to show that V_i is convex for every i. Let $x_i, x_i' \succ_i x_i^*$. We need to show that

$$\forall 0 < \lambda < 1, \lambda x_i + (1-\lambda) x_i' \succ_i x_i^*$$

W.L.O.G., suppose $x_i \succeq_i x_i'$. Convexity transitivity of $\succ_i$ imply

$$\lambda x_i + (1-\lambda) x_i' \succeq_i x_i' \succ_i x_i^*$$

■

Next, $V \cap Y + \sum_i \omega_i = \emptyset$ (otherwise, there would be a production pattern that would leave each consumer with a strictly preferred consumption bundle).

Next, by the Separating Hyperplane Theorem, there exists $p \neq 0$ with $px \geq c$ for all $x \in V$, and $py \leq c$ for all $y \in Y + \sum_i \omega_i$. By the assumption of nonsatiation, $p >> 0$ (not only $p \neq 0$).

Next, $p \sum_i x_i^* = p(\sum_i \omega_i + \sum_j y_j^*) = c$. This follows from the fact that x^* belongs to two sets: V and $Y + \sum_i \omega_i$.

Lastly, we need to show that for every i, a strictly preferred bundle is not affordable at p. Suppose that $x_{\text{Oz}} \succ_{\text{Oz}} x_{\text{Oz}}^c$. Clearly,

$$x_{\text{Oz}} + \sum_{i \neq \text{Oz}} x_i^* \in V$$

Hence,

$$p \left(x_{\text{Oz}} + \sum_{i \neq \text{Oz}} x_i^* \right) > c$$

Hence, $px_{\text{Oz}} > px_{\text{Oz}}^*$. ■

2.7 Social Welfare Functions & Pareto Optimality

The issues:

- The 2nd Welfare Theorem facilitates the calculation of competitive equilibria (especially in dynamic models)
- The social welfare function facilitates finding Pareto-optimal allocations (hence competitive equilibria)

DEFINITION 2.9 *The **utility possibility set** is:*

$$U \stackrel{def}{=} \{(u_1, \ldots, u_I) \in \mathbb{R}^I : u_i \le u_i(x_i) \quad (x, y) \textit{ is a feasible allocation}$$

*The **Pareto frontier** is the set*

$$UP \stackrel{def}{=} \{(u_1, \ldots, u_I) \in U : \textit{ s.t. } \nexists (u_1', \ldots, u_I') \in U \textit{ with } u_i' \ge u_i \ \ \forall i \quad \textit{and } u_i' > u_i \textit{ some } i$$

Proposition 2.3 *A feasible allocation (x, y) is Pareto optimum if and only if $(u_1(x_1), \ldots, u_I(x_I)) \in$ UP.*

Proof. Suppose that $(u_1(x_1), \ldots, u_I(x_I)) \notin$ UP. Then,

$$\exists (u_1', \ldots, u_I') \in U \text{ with } u_i' \ge u_i \ \ \forall i \quad \text{and } u_i' > u_i \text{ some } i$$

which is feasible since is $\in U$. Hence, (x', y') is feasible and Pareto dominates (x, y). A contradiction.

Suppose now that (x, y) is not PO. Then, there exists a feasible allocation (x', y') such that $u_i(x_i) \ge u_i(x_i)$ $\forall i$ (and $>$ for some i). Hence, $(u_1(x_1), \ldots, u_I(x_I)) \notin$ UP. ■

DEFINITION 2.10 *A **social welfare function** is $W : \mathbb{R}_+^I \mapsto \mathbb{R}_+$, such that $W(u_1, \ldots, u_I)$ is nondecreasing in each u_i. Social Welfare function is **linear** if*

$$W(u_1, \ldots, u_I) = \sum_{i=1}^{I} \lambda_i u_i \quad \textit{for some } (\lambda_1, \ldots, \lambda_I), \ \lambda_i \ge 0$$

Remark: sufficient to consider $\sum_{i=1}^{I} \lambda_i = 1$.

Proposition 2.4 *Consider the maximization problem*

$$\max_{(u_1, \ldots, u_I) \in U)} \sum_{i=1}^{I} \lambda_i u_i \quad \lambda_i > 0 \ \forall i.$$

If $(u_1^*, \ldots, u_I^*)$ is a solution for this problem then $(u_1^*, \ldots, u_I^*) \in$ UP (i.e., it is the utility vector of a PO allocation).

Proof. Suppose not. Then,

$$\exists (u_1, \ldots, u_I) \in U \text{ where } u_i \ge u_i^* \ \forall i, \text{ and } u_i \ u_i^* \text{ some } i$$

Hence,

$$\sum_{i=1}^{I} \lambda_i u_i > \sum_{i=1}^{I} \lambda_i u_i^*$$

implying that $(u_1^*, \ldots, u_I^*)$ is not the solution of the social welfare maximization problem.

2.8 The Core

2.8.1 Definitions

- Looks at markets with large number of traders with no Walrasian auctioneer
- more strategic traders (can form coalitions)
- Provides more intuition on the relationship between competitive equilibria and welfare.
- Look at pure exchange (see text for production)
- I consumers, $i \in N \stackrel{\text{def}}{=} \{1, 2, \ldots, I\}$
- each i has endowment $\omega_i \in \mathbb{R}_+^L$
- $\succeq_i$ is:
    1. strictly monotone
    2. strictly convex
    3. continuous
- Allocation $x = (x_1, \ldots, x_I) \in \mathbb{R}_+^{L \times I}$ is *feasible* if $\sum_i^I x_i = \sum_i^I \omega_i$

Definition 2.11 *1. A **coalition** is $S \subseteq N$ ($S \neq \emptyset$). That is, a nonempty subset of consumers.*

*2. A Coalition is called **grand coalition** if $S = N$.*

Definition 2.12 *A coalition $S \subseteq N$ improves upon, or **blocks** a feasible allocation $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_i, \ldots, \hat{x}_I)$ if $\forall i \in S$, there exist consumption vectors $x_i \geq 0$ such that*

1. $x_i \succ_i \hat{x}_i$

2. $\sum_{i \in S} x_i = \sum_{i \in S} \omega_i$

Remark 1: requiring strict preference (although can prove that under strict monotonicity (assumed) we can always compensate those who are indifferent with those who are strictly better off in the same coalition (see homework assignment).
Remark 2: Example: Ex Russian states, California, Jerusalem Suburbs, Punjab region.

Definition 2.13 *1. A feasible allocation $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_i, \ldots, \hat{x}_I)$ has the **core property** if there does not exist a coalition $S \subseteq N$ that blocks $\hat{x}$.*

*2. The **core** is the set of all feasible allocation that have the core property.*

Remark: Show now an Edgeworth Box with contract curve and core allocations with 2 consumers.

2.8.2 The core and efficiency, and competitive equilibria

Proposition 2.5 *Every allocation in the core is Pareto efficient*

Proof. Suppose not. Then, an allocation can be blocked by the grand coalition. Hence, it is not in the Core. A contradiction. ∎

Proposition 2.6 *A competitive equilibrium allocation has the core property*

Proof. (resembles very much the proof of the First Welfare Theorem).
Let $\hat{x} = (\hat{x}_1, \dots, \hat{x}_i, \dots, \hat{x}_I)$ be a competitive equilibrium corresponding to the price vector $\hat{p}$.

BWOC, suppose that there exists a blocking coalition $S \subseteq N$. Then, there exist consumption vectors $\{x_i\}_{i \in S}$ such that $x_i \succ_i \hat{x}_i$ for all $i \in S$.

Hence, $\hat{p}x_i > \hat{p}\omega_i$, $\forall i \in S$.
Summing up over $i \in S$,

$$\hat{p}\left(\sum_{i \in S} x_i\right) > \hat{p}\left(\sum_{i \in S} \omega_i\right)$$

contradicting $\sum_{i \in S} x_i = \sum_{i \in S} \omega_i$.
Hence, S does not block $\hat{x}$. ∎

2.8.3 Growing economy: replication of an economy

- Show picture p.655 showing how a replica of 2×2 can shrink the core
- This analysis is valid only for equal number of consumer types (i.e., N of each type, thus, called N-replica).
- Let there be H types
- hence, total number of consumers in the economy is $N \times H$

Definition 2.14 *Consumers k and ℓ are of the **same type** if $\succeq_k = \succeq_\ell$ and $\omega_k = \omega_\ell$.*

Definition 2.15 *An allocation is said to satisfy **equal-treatment property** is all consumers of the same type receive the same consumption bundles.*

Proposition 2.7 *Let x be in the core of the N-replica of an economy with H types. Then, x satisfies the equal-treatment property. Formally,*

$$x_{hm} = x_{hn} \quad \forall n, m \in \{1, \dots, N\} \qquad \text{and } h \in \{1, \dots, H\}.$$

Proof. Suppose not. Then suppose that for type $h = 1$, $x_{1m} \neq x_{1n}$, $n \neq m$.

Since there is no equal treatment, suppose that for every type h, consumer $h1$ is worst-treated (underdog). Formally,

$$x_{hn} \succeq_h x_{h1} \quad \forall n \neq 1.$$

For each type h define the average bundle of type h as

$$\bar{x}_h \stackrel{\text{def}}{=} \frac{\sum_n x_{hn}}{N}.$$

By convexity of preferences,

$$\bar{x}_h \succeq_h x_{h1} \quad \forall h, \quad \text{and} \quad \bar{x}_1 \succ_1 x_{11}.$$

Consider the coalition $S \stackrel{\text{def}}{=} \{11, \ldots, h1, \ldots, H1\}$ (i.e., all worst-treated consumers of each type). Clearly, each type can attain the bundle $\bar{x}_h$ (preferred).
We need to demonstrate feasibility:

$$\sum_h \bar{x}_h = \frac{\sum_h (\sum_n x_{hn})}{N} = \sum_h \left(\frac{\sum_n x_{hn}}{N} \right) \sum_h \omega_h.$$

■

Proposition 2.8 *Let C_N denote the set of core allocations for the N-replica. Then,*

$$C_{N+1} \subseteq C_N, \quad \forall N.$$

Proof. Take an allocation outside the core, $x \notin C_N$. The same coalition that blocked x can block x in a larger replica $N+1$. ■

Proposition 2.9 *Let $N \to \infty$. If $\hat{x} \in C_N$ for all N, then $\hat{x}$ is a competitive equilibrium.*

Proof.

- Take x which is not CE.
- Show, that it can be blocked for N sufficiently large
- Let x be PO (otherwise be blocked by the grad coalition)
- let p be a supporting price vector by the 2nd welfare theorem
- Since x is not CE, there exists a consumer (say all type $h = 1$) for which $p(x_1 - \omega_1) > 0$.
- Get a coalition with one of type $h = 1$ out. Then each coalition member share equally the net transfer to 1:
$$x'_h = x_h + \frac{x_1 - \omega_1}{(N-1) + N(H-1)}$$
- can show feasibility:
$$(N-1)x'_1 + Nx'_2 + \cdots + Nx'_H = (N-1)\omega_1 + N\omega_2 + \cdots + N\omega_H$$
- Are all blocking coalition members better off blocking?
- Yes, since
$$p(x_1 - \omega_1) = \sum_{\ell=1}^{L} p_\ell(x_{1,\ell} - \omega_{1,\ell}) > 0 \Longrightarrow \sum_{\ell=1}^{L} \frac{\Delta U_h}{\Delta x_\ell}(x_{1,\ell} - \omega_{1,\ell}) > 0$$

Lecture 3
Externalities

3.1 Major Issues

- 1st Welfare Theorem: CE $\Longrightarrow$ PO
- Basic assumption: (a) no externalities, (b) complete markets
- 1st Welfare Theorem may fail under (a) or (b). We call it *market failure.*
- Consumption externality: If $\vec{x}_i$ is consumption bundle of consumer i, then a consumption externality occurs when $u_i(\vec{x}_i, \vec{x}_j)$ for some $j \neq i$.
- Note: Text defines externality as an action h_j that enters i's utility: $u_i(\vec{x}_i, h_j)$.
- Examples for production externalities: Noise, appearance (personal and property), profession (neighborhoods), cooking (smell), cleaning.
- Production externality: Let $\vec{z}_i$ be a vector of inputs of firm i, and let y_i be the output level. Then, a production externality occurs if $y_i = F(\vec{z}_i; y_j)$ for some firm $j \neq i$.
- Note: Text defines externality as an action h_j that enters firm j's production function, i.e., $y_i = F(\vec{z}_i; h_j)$
- Can mix production and consumption externalities.
- Two types of externalities: *depletable* (garbage); *nondepletable* (polluted air or rain)
- *Pecuniary externality:* action of one agents affect profitability of other agent exists in *any* environment. However, competitive behaviour generates the 1st-welfare theorem.
- Policy question: how can we correct market failure?
- Answer: (i) Taxes (Pigou) , (ii) property right + bargaining (Coase), and (iii) establish a market (Arrow).

3.2 Example

- Airport choosing the number of landings x
- Housing developer choosing # houses to build, y

$$\pi^A = 48x - x^2 \quad \text{and} \quad \pi^D = 60y - y^2 - xy$$

3.2.1 No regulations, no communication

Airport: $0 = 48 - 2x \Longrightarrow\ x = 24$.
Developer: $0 = 60 - 2y - x \Longrightarrow\ y = 18$.
Profit levels: $\pi^A = 576$; $\pi^D = 324$; $\Pi \stackrel{\text{def}}{=} \pi^A + \pi^D = 900$.

3.2.2 Pareto- (industry) optimal allocation

$$\max_{x,y} \Pi = 48x - x^2 + 60y - y^2 - xy$$

$$\begin{aligned} 0 &= \frac{\partial \Pi}{\partial x} = 48 - 2x - y \\ 0 &= \frac{\partial \Pi}{\partial y} = 60 - 2y - x \end{aligned}$$

Hence, $y^* = 24$ and $x^* = 12$.

$$\pi^A = 432; \quad \pi^D = 576; \quad \Pi^* = 1008$$

3.2.3 "Strict prohibition:" developer has all property right (bargaining not allowed)

Hence, no planes, $x = 0$, $\pi^A = 0$.

$$\pi^D = 60y - y^2 - 0 \Longrightarrow\ 0 = 60 - 2y \Longrightarrow\ y = 30 \Longrightarrow\ \pi^D = 30^2 = 900 < \Pi^*.$$

Hence, this mechanism does not support PO.

3.2.4 Airport is liable for all damages (bad mechanism!)

Airport must pay xy to developer (compensation for the damage).

$$\max_x \pi^A = 48x - x^2 - xy = 48x - x^2 - 30x \Longrightarrow\ 0 = 18 - 2x \Longrightarrow\ x = 9$$

$$\pi^D = 60y - y^2 - xy + xy \Longrightarrow\ 0 = 60 - 2y \Longrightarrow\ y = 30$$

Hence,

$$\pi^A = 81; \quad \pi^D = 900; \quad \Pi = 981 < \Pi^*$$

This mechanism also does not support PO since the developer will over-produce to increase the compensation.

3.2.5 Coase equilibrium (bargaining after property rights are assigned to the airport): developer bribes the airport

The developer decides on x subject to leaving $\pi^A = 576$.
Level of the bribe $= 576 - (48x - x^2)$.

$$\max_{x,y} \pi^D = 60y - y^2 - xy - [576 - (48x - x^2)]$$

$$\begin{aligned} 0 &= \frac{\partial \pi^D}{\partial y} = 60 - 2y - x \\ 0 &= \frac{\partial \pi^D}{\partial x} = -y + 48 - 2x \end{aligned}$$

Hence, $y = 24 = y^*$ and $x = 12 = x^*$. *Pareto-allocation!!!*

Coase Theorem: if agents can bargain, then optimality is restored regardless of property rights assignment.

3.2.6 Coase equilibrium (bargaining property rights are assigned to the Developer): Airport bribes the developer to allow him to increase x

Airport has to leave the developer with at least $\pi^D \geq 900$ (see Subsection 3.2.4).
The compensation is $900 - (60y - y^2 - xy)$. Hence,

$$max_{x,y}\pi^A = 48x - x^2 - [900 - (60y - y^2 - xy)].$$

Yielding the same PO allocation with different profit levels.
Remark: Why property rights are needed? Answer: to define the reservation payoffs of the players before bargaining starts.

3.2.7 Optimal tax

- It is essential that the tax will be levied on the externality-producing activity directly!!!
- i.e., do not tax the food sold in airport restaurants, since consumers will bring their own food, but will not reduce travel. A tax on gas, will lead to refueling abroad.
- Exception: when the taxed item is a perfect complement to the externality activity.
- However, even if you tax the activity only, flights may switch to airports nearby in a different country.

Which tax on airport will restore optimality? i.e., $x^* = 12$, hence, $y^* = 24$.

$$\pi^A = 48x - x^2 - tx \Longrightarrow \; 0 = 48 - 2x - t^* \Longrightarrow t^* = 24$$

The rest follows.

3.3 Appendix: Quasi Linear Utility Functions

Consider a consumer who has preferences for two items: money (m) and the consumption level (Q) of a certain product, which he can buy at a price of p per unit.

$$U(Q, m) \equiv \sqrt{Q} + m. \tag{3.1}$$

Consumer is endowed with a fixed income of I to be spent on the product or to be kept by the consumer. Then, if the consumer buys Q units of this product, he spends pQ on the product and retains an amount of money equals to $m = I - pQ$.

$$\max_{Q} U(Q, I - pQ) = \sqrt{Q} + I - pQ. \tag{3.2}$$

The first-order condition is given by $0 = \partial U/\partial Q = 1/(2\sqrt{Q}) - p$, and the second order by $\partial^2 U/\partial Q^2 = -1/(4Q^{-3/2}) < 0$, which constitutes a sufficient condition for a maximum.

$$p(Q) = \frac{1}{2\sqrt{Q}} = \frac{Q^{-1/2}}{2}. \tag{3.3}$$

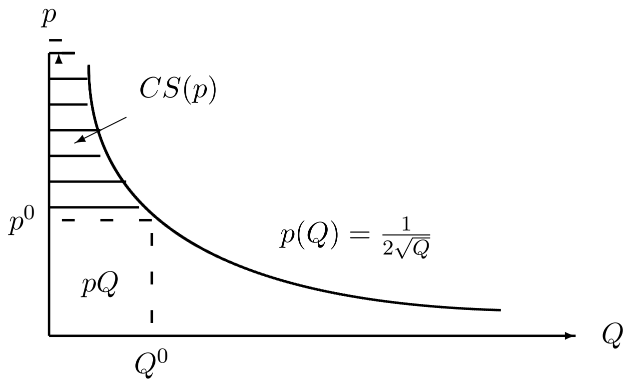

Figure 3.1: Inverse demand generated from a quasi-linear utility function

Proposition 3.1 *If a demand function is generated from a quasi-linear utility function, then the area marked by $CS(p)$ in Figure 3.1 measures exactly the utility the consumer gains from consuming Q^0 units of the product at a market price p^0.*

Proof. The area $CS(p)$ in Figure 3.1 is calculated by

$$\begin{aligned} CS(p) &\equiv \int_0^{Q^0} \left(\frac{1}{2\sqrt{Q}}\right) dQ - p^0 Q^0 \\ &= \sqrt{Q^0} - p^0 Q^0 = U(Q^0, I - p^0) \end{aligned} \tag{3.4}$$

■

3.4 General Formulation Using Q-L Utility Functions

- 2 consumers, $i = 1, 2$
- L traded goods, $p \in \mathbb{R}^L$ price vector (fixed to consumers)
- $h \in \mathbb{R}_+$ is action taken by consumer 1
- Let $x_i \stackrel{\text{def}}{=} (x_{1i}, x_{Li})$ consumption vector of consumer i
- $u_i(x_{1i}, \dots, x_{Li}, h)$, utility of i, $i = 1, 2$
- Assume externality on consumer 2: $\partial u_2(x_{12}, \dots, x_{L2}, h)/\partial h \neq 0$

- We can define a consumer's "indirect" utility as a function of the externality:

$$V_i(p, w_i, h) \stackrel{\text{def}}{=} \max_{x_i \geq 0} u_i(x_i, h) \quad \text{s.t. } p \cdot x_i \leq w_i$$

- focus on quasi-linear (indirect) utility: $V_i(p, w_i, h) \stackrel{\text{def}}{=} \phi_i(p, h) + w_i$, or, since p are fixed, $\phi_i(h) + w_i$, $\phi_i'' < 0$.
- Remark: same math works for the firm, where $\pi_i(p, h)$ behave similarly.

3.4.1 Nonoptimality of the competitive outcome

In competition, consumer 1 (who generates the externality) chooses h to

$$\max_h \phi_1(h) + w_1 \implies 0 = \phi_1'(h^e)$$

Social Planner solves

$$\max_h \phi_1(h) + w_1 + \phi_2(h) + w_2 \implies 0 = \phi_1'(h^*) + \phi_2'(h^*) \implies \phi_1'(h^*) = -\phi_2'(h^*) \text{ for } h^* > 0$$

Two cases:

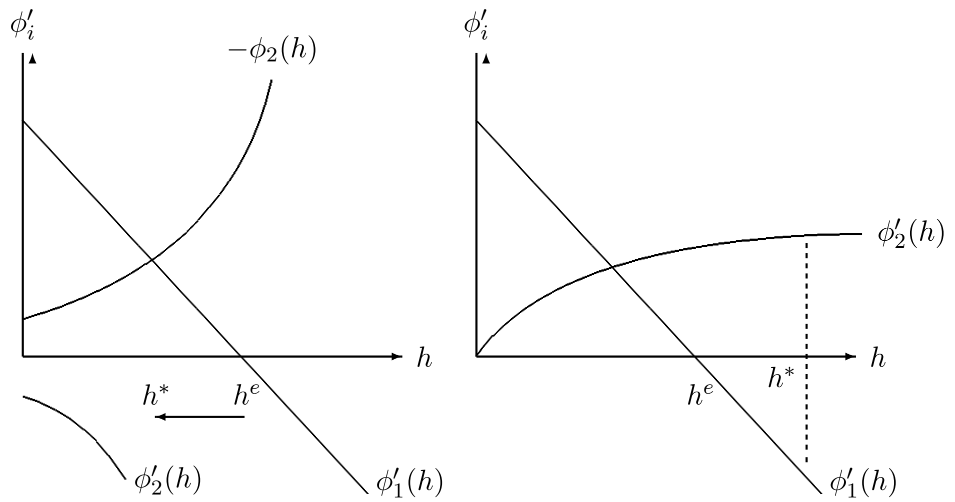

Figure 3.2: Social optimum vs. the competitive outcome. *Left:* Negative externality. *Note:* There is no implied meaning to the intercepts and to the linearity of one of the marginal utilities. *Right:* Positive externality

3.4.2 Restoring optimality via Pigouvian taxation

Negative externality: Choose t_h to that h^* will be chosen by agent 1 so

$$h^* = \arg\max_h \phi_1(h) - t_h h$$

implying government should set $t_h^* = -\phi_2'(h^*)$.

Positive externality: Choose s_h to that h^* will be chosen by agent 1 so

$$h^* = \arg\max_h \phi_1(h) + s_h h$$

implying government should set $s_h^* = \phi_2(h^*)$.

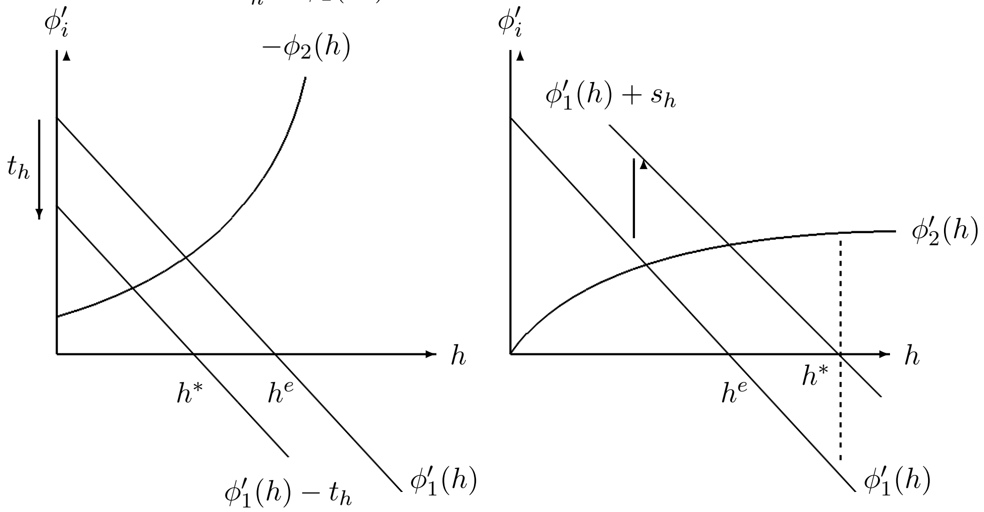

Figure 3.3: Social optimum vs. the competitive outcome. *Left:* Negative externality. *Right:* Positive externality

Externality-use tax vs. externality-reduction subsidy Government offers a subsidy r (reduction) for every unit of h reduced below h^e. Calculate r^*!
Agent 1 solves

$$\max_h \phi_1(h) + r(h^e - h)$$

Now, set $r^* \equiv t_h^*$, hence, a subsidy for reduction is equal to optimal tax plus lump-sum subsidy of $t_h^* \times h^e$.

3.4.3 Coase: property rights + bargaining

- Before bargaining, property rights should be well defined
- We'll focus on 2 extreme property rights
- Final distribution of payoffs depend on the underlined bargaining process (ad-hoc assumption).
- We'll focus on take-it-or-leave-it offer

Property rights assigned to agent 2 (externality-free environment) Assume negative externality.
Agent 2 states: pay me $\$T$ and I'll allow you to generate h-level of externality.

Agent 2 chooses a pair (h, T) to leave agent 1 with a reservation utility $\geq \phi_1(0)$.

$$\max_{h,T} \phi_2(h) + T \quad \text{s.t. } \phi_1(h) - T \geq \phi_1(0)$$

Monotonicity implies that the constraint is binding: $T = \phi_1(h) - \phi_1(0)$.

$$\max_{h \geq 0} \phi_2(h) + \phi_1(h) - \phi_1(0) \Longrightarrow \phi_1'(h^*) = -\phi_2'(h^*)$$

Property rights assigned to agent 1 (pollution allowed) Agent 2 states: I'll pay you \$$T$ to reduce from h^e to $h < h^e$.
Now, reservation utility is $\phi_1(h^e)$.

$$\max_{h,T} \phi_2(h) - T \quad \text{s.t. } \phi_1(h) + T \geq \phi_1(h^e)$$

Monotonicity implies that the constraint is binding: $T = \phi_1(h^e) - \phi_1(h)$.

$$\max_{h \geq 0} \phi_2(h) - [\phi_1(h^e) - \phi_1(h)] \Longrightarrow \phi_1'(h^*) = -\phi_2'(h^*)$$

3.4.4 Missing markets

- Arrow (1962): Optimality is restored when a market for the externality is established.
- somewhat unrealistic in a market with many agents
- In this market, each agent pays p_h for the right to generate externality (so, in the present setup, agent 2 has the right for externality-free environment).
- hence, agent 2 is the seller.
- competitive market, price-taking behavior

Agent 1 takes p_h as given and chooses h_1 (to buy) to

$$\max_{h_1} \phi_1(h_1) - p_h h_1 \Longrightarrow \phi_1'(h_1) = p_h$$

Agent 2 takes p_h as given and chooses h_2 (to sell) to

$$\max_{h_2} \phi_2(h_2) + p_h h_2 \Longrightarrow \phi_2'(h_2) = -p_h$$

Hence, in a competitive equilibrium, $\phi_1'(h_1) = p_h^e = -\phi_2'(h_2)$

Example for missing markets:

$$\phi_1(h_1) = \sqrt{h_1} \quad \text{and} \quad \phi_1(h_2) = -(h_2)^2$$

$$\max_{h_1} \sqrt{h_1} - p_h h_1 \Longrightarrow p_h = \frac{1}{2\sqrt{h_1}}.$$

$$\max_{h_2} p_h h_2 - (h_2)^2 \Longrightarrow p_h = 2h_2$$

Hence, $\hat{h} = 1/2$ and $\hat{p}_h = 1$.

Lecture 4
Public Goods

- Defined as a commodity/service for which use of a unit by one agent does not preclude its use by other agents
- Examples: bridges, radio, TV, swimming in the lake, breathing, knowledge, know-how.
- I consumers, G = amount of public good provided (to everybody)
- g_i= monetary contribution to the public good by consumer i
- Linear production of public good $G = \sum_{i=1}^{I} g_i$
- x_i = consumption of private good by consumer i (numeraire $p_x = 1$), w_i=income, hence $x_i = w_i - g_i$.
- $$u_i(G, x_i) = u_i\left(\sum_{i=j}^{I} g_j, w_i - g_i\right)$$
- P.O. allocation: can be calculated as follows: Let $0 \le \alpha_i \le 1$, with $\sum_{i=1}^{I} \alpha_i = 1$

$$W \stackrel{\text{def}}{=} \max_{\substack{g_1,\dots,g_I \\ x_1,\dots,x_I}} \sum_{i=1}^{I} \alpha_i u_i\left(\sum_{j=1}^{I} g_j, w_i - g_i\right)$$

Let $\alpha_i > 0$.

$$0 = \frac{\partial W}{\partial g_k} = \sum_{i=1}^{I} \alpha_i \frac{\partial u_i}{\partial G}\frac{\partial G}{\partial g_k} - \alpha_k \frac{\partial u_k}{\partial x_k}, \quad k = 1, \dots, I$$

Hence,

$$\alpha_i \frac{\partial u_i}{\partial x_i} = \alpha_j \frac{\partial u_j}{\partial x_j} \quad \forall\ i, j = 1, \dots, I$$

Dividing the RHS for each k

$$1 = \frac{\sum_{i=1}^{I} \alpha_i \frac{\partial u_i}{\partial G}}{\alpha_k \frac{\partial u_k}{\partial x_k}} = \frac{\alpha_1 \frac{\partial u_1}{\partial G}}{\alpha_1 \frac{\partial u_1}{\partial x_1}} + \cdots + \frac{\alpha_I \frac{\partial u_I}{\partial G}}{\alpha_I \frac{\partial u_I}{\partial x_I}} = \sum_{i=1}^{I} \frac{\frac{\partial u_i}{\partial G}}{\frac{\partial u_i}{\partial x_i}}.$$

4.1 Quasi linear case: P.O. allocation

- $u_i = \phi_i(Q) + x_i$, where $\phi_i' > 0$ and $\phi_i'' < 0$ (rising and concave)
- single public good producer, $c(Q)$, $c' > 0$, $c'' > 0$.

$$\max_Q \sum_{i=1}^{I} \phi_i(Q) - c(Q) \Longrightarrow \sum_{i=1}^{I} \phi_i'(Q) = c'(Q) \quad \text{for } Q > 0$$

This is *Samuelson's condition.*
i.e., sum of $MU_i = MC$ (contrary to private goods where each MU_i equals to MC.)

4.2 Private Provision of Public Good: Perfect competition case

- Is there a market failure? In which direction?
- q_i = amount of public good demanded by consumer i
- $Q = \sum_{i=1}^{I} q_i$ = actual consumption
- $c(Q)$ production cost, $c' > 0$, $c'' > 0$ (single producer)

Producer supplies Q units to a market for a given price p^e as to

$$\max_Q \pi = pQ - c(Q) \Longrightarrow p = c'(Q)$$

Each consumer takes p^e and $\sum_{j\neq i}^{I} q_j$ as given and chooses q_i to

$$\max_{q_i} \phi_i \left(q_i + \sum_{j\neq i}^{I} q_j \right) - p^e q_i \Longrightarrow \phi_i' \left(q_i + \sum_{\substack{j=1 \\ j\neq i}}^{I} q_j \right) = p^e \quad \text{if } q_i > 0$$

Hence, in equilibrium, demand equals supply

$$\phi_i' \left(\sum_{i=1}^{I} q_i^e \right) = \phi_i'(Q^e) = p^e = c'(Q^e)$$

Hence,

$$\sum_{i=1}^{I} \phi_i'(Q^e) > c'(Q^e) \Longrightarrow Q^e < Q^* \text{ (since } c'' > 0 \text{ and } \phi_i'' < 0)$$

Why is there a market failure?

1. each consumer does not pay for his/her entire consumption level.
2. *free-rider effect*
3. Equilibrium price is "too low" for producers
4. Equilibrium price is "too high" for consumers (sum exceeds producer's price)
5. hence, a *Lindahl equilibrium* must "fix" both prices

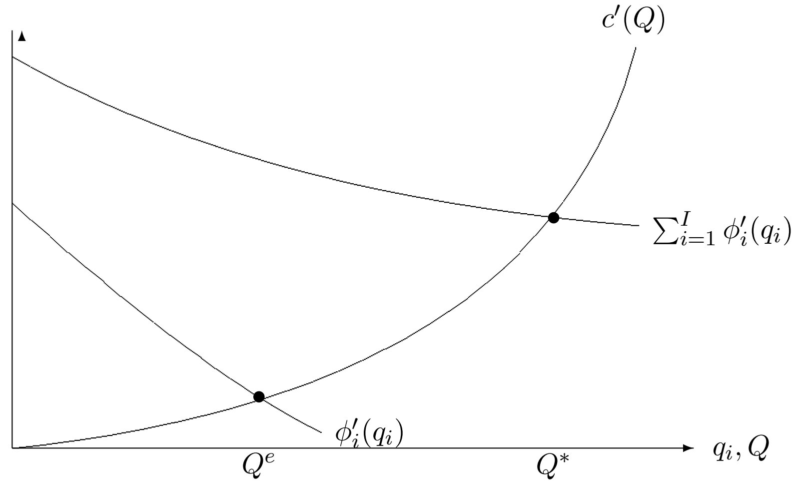

Figure 4.1: Underprovision of privately-provided public good

4.3 Lindahl Equilibrium

- Competitive equilibrium $\Longrightarrow$ producer's price is "too low," and consumer price is "too high"
- Lindhal prices (taxes) do support PO as a CE
- Suppose that each consumer pays for his/her *entire* consumption of the public good at the level he chooses, x_i at a given price p_i^L
- and the firms supplying Q and selling it separately to the I consumers

Consumer i solves

$$\max_{q_i} \phi_i(q_i) - p_i^L q_i \Longrightarrow \phi_i'(q_i) = p_i^L \Longrightarrow \sum_{i=1}^{I} \phi_i'(q_i^L) = \sum_{i=1}^{I} p_i^L$$

Given p_i, $i = 1, \ldots I$, the producer sells the same amount Q to each of the I consumers as to

$$\max_{Q \geq 0} \pi(Q) = \left(Q \sum_{i=1}^{I} p_i^L \right) - c(Q) \Longrightarrow \sum_{i=1}^{I} p_i^L = c'(Q^L)$$

Hence,

$$\sum_{i=1}^{I} \phi_i'(q_i^L) = \sum_{i=1}^{I} p_i^L = c'(Q^L)$$

Hence, $Q^L = Q^*$.
Remarks:

1. Intuition: each cosnumer consumes his true valuation level given price
2. Note that each consumer actually consumes $Q^L > q_i$.
3. Exclusion of consumers is necessary. Otherwise, they will not buy.

Lecture 5
Asymmetric Information

5.1 Signaling

5.1.1 Example of Milgrom & Roberts Limit-Pricing Problem

- 2 periods, $t = 1, 2$.
- demand each period $p = 10 - Q$.
- Stage 1: firm 1 chooses q_1^1.
- Stage 2: firm 2 chooses to enter or not
- Stage 2: Assumption: Entry occurs: Cournot; Does not: Monopoly
- Firm 2: $c_2 = 1$; $F = 9$ = entry cost
- Firm 1:

$$c_1 = \begin{cases} 0 & \text{with probability } 0.5 \\ 4 & \text{with probability } 0.5. \end{cases}$$

Incumbent's cost:	Firm 2 (potential entrant)			
	ENTER		DO NOT ENTER	
Low ($c_1 = 0$)	$\pi_1^c(0) = 13$	$\pi_2^c(0) = -1.9$	$\pi_1^m(0) = 25$	$\pi_2 = 0$
High($c_1 = 4$)	$\pi_1^c(4) = 1$	$\pi_2^c(4) = 7$	$\pi_1^m(4) = 9$	$\pi_2 = 0$

Table 5.1: Profit levels for $t = 2$ (depending on the entry decision of firm 2). *Note:* All profits are functions of the cost of firm 1 (c_1); π_1^m is the monopoly profit of firm 1; π_i^c is the Cournot profit of firm i, $i = 1, 2$.

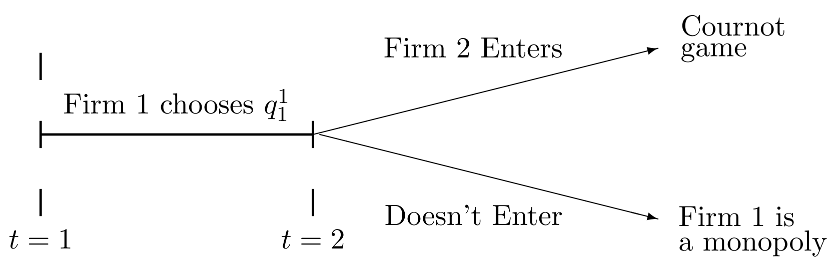

Figure 5.1: Two-period, signaling, entry-deterrence game. *Note:* Add stage $t = 0$, where nature selects c_1.

Solving the game assuming signaling is not implemented

$$E\pi_2^c = \frac{1}{2}\pi_2^c(0) + \frac{1}{2}\pi_2^c(4) = \frac{1}{2}(-1.9) + \frac{1}{2}7 > 0,$$

Hence, firm 2 will enter.

Given that entry occurs at $t = 2$, firm 1 should play monopoly at $t = 1$.

$$q_1^1(4) = 5 \ \text{ and therefore earn } \ \pi_1(4) = \pi_1^m(4) + \pi_1^c(4) = 9 + 1 = 10. \tag{5.1}$$

Solving the game assuming a low-cost incumbent

If $c_1 = 0$, $\pi_2^c(0) < 0$, hence, no entry (under perfect information).

Proposition 5.1 *A low-cost incumbent would produce $q_1^1 = 5.83$, and entry will not occur in $t = 2$.*

Sketch of Proof. A high-cost incumbent would not produce $q_1^1 = 5.83$ since

$$9.99 = (10 - 5.83) \times 5.83 - 4 \times 5.83 + \pi_1^m(4) < \pi_1^m(4) + \pi_1^c(4) = 9 + 1 = 10. \tag{5.2}$$

That is, a high-cost incumbent is better off playing a monopoly in the first period and facing entry in the second period than playing $q_1^1 = 5.83$ in the first period and facing no entry in $t = 2$.

Hence, we have to show that deterring entry by producing $q_1^1 = 5.83$ yields a higher profit than accommodating entry and producing the monopoly output level $q_1^1 = 5$ in $t = 1$.

$$\pi_1(0)|_{q_1^1=5} = 25 + 13 = 38 < 49.31 = (10 - 5.83) \times 5.83 + 25 = \pi_1(0)|_{q_1^1=5.83},$$

hence, a low-cost incumbent will not allow entry and will not produce $q_1^1 < 5.83$. ■

5.1.2 Quality signaling

- A monopoly firm knows the quality of the brand it sells
- consumers are unable to learn the brand's quality prior to the actual purchase (Nelson's *experience good*)
- Our goal is to demonstrate that a monopoly firm can signal the quality it sells by choosing a certain price and by imposing a quantity rationing
- A continuum of identical consumers. normalize the number of consumers to equal 1.
- Each consumer buys, at most, one unit
- For a given price denoted by p, the utility function of each consumer is given by

$$U \equiv \begin{cases} H - p & \text{if the brands happens to be of high quality} \\ L - p & \text{if the brands happens to be of low quality} \\ 0 & \text{if he does not purchase the product.} \end{cases}$$

- c_H the unit production cost of the monopoly if it is a high-quality producer

- c_L if it is a low-quality one, where $c_H > c_L \geq 0$.
- Assumption:
  1. The monopolist is a high-quality producer (Nature drawn)
  2. $H > L > c_H$.
- Monopoly's 2D strategy ($p \in [0, \infty)$) and ($q \in [0, 1]$)
- How can a high-quality producer convince the consumers that he or she does not cheat by selling a low-quality brand for a high price?

Proposition 5.2 *There exists a pair of a price and a quantity level that convinces consumers (beyond all doubts) that the brand they buy is a high-quality one. Formally, if the monopolist sets*

$$p^m = H \quad \textit{and} \quad q^m = \frac{L - c_L}{H - c_L},$$

Then,

1. *consumers can infer that the brand is of high quality,*
2. q^m *consumers will purchase the product and* $(1 - q^m)$ *consumers will not purchase the brand due to the lack of supply.*

Proof.

- Must show that a low-quality producer would not choose p^m and q^m
- If the monopolist were a low-quality producer then set $p = L$ and make a profit of $\pi^L(L, 1) = 1(L - c_L)$.
- A low-quality monopoly cannot profit from choosing p^m and q^m since,

$$\pi^L(p^m, q^m) = (p^m - c_L)q^m = (H - c_L)\frac{L - c_L}{H - c_L} = L - c_L = \pi(L, 1).$$

■

Cost if revealing information:

$$\pi^H(p^m, q^m) = p^m q^m = (H - c_H)\frac{L - c_L}{H - c_L} < H - c_H.$$

Is it profitable?

$$(H - c_H)q^m = (H - c_H)\frac{L - c_L}{H - c_L} \geq (L - c_H)1. \tag{5.3}$$

Cross-multiplying (5.3) yields that this inequality always holds, since $H > L > c_H > c_L$.

5.1.3 Signaling via warranty

- Two brands, unknown quality
- High quality producer: $\rho_H = \Pr(\text{operative})$
- Law quality producer: $\rho_L = \Pr(\text{operative})$
- $0 < \rho_L < \rho_H < 1$
- Bertrand competition: No warranty implies $p_i^{\text{NW}} = c$ and $\pi_i^{\text{NW}} = 0$ $i = H, L$.
- Utility for a given operational probability:

$$U = \begin{cases} \rho V - p & \text{No warranty} \\ V - p & \text{With} \\ 0 & \text{No purchase} \end{cases}$$

Warranty as a signal

Proposition 5.3 *Let $V > c$. The high-quality producer can push the low quality producer out of the market by setting $p^W = c/\rho_L$ and providing a warranty. In this case the consumer will buy only the more reliable product, and the high-quality producer will make a strictly positive profit.*

Proof.

- A LQ producer will not find it profitable to sell with a warranty at this price since

$$\pi_L^W(p^W) = p^W - \frac{c}{\rho_L} = 0.$$

- The high-quality producer makes an above zero profit.

$$\pi_H^W = p^W - \frac{c}{\rho_H} = \frac{c}{\rho_L} - \frac{c}{\rho_H} > 0.$$

- the utility of a consumer buying the more reliable product exceeds the utility of buying the less reliable product without a warranty even if the less reliable product has the lowest possible price, c, since

$$U_H^W = V - p^W = V - \frac{c}{\rho_L} > U_L^{NW} = \rho_L V - c.$$

■

5.1.4 Labor market signaling: Separating vs. pooling equilibria

- Two types of workers (2 productivities): θ^H and θ^L.
- $\lambda = \Pr(\theta = \theta^H) \in (0, 1)$.
- $e =$ level of education (i.e., degree completed)
- $c(e, \theta) =$ cost of achieving education level of e

- The function $c(e, \theta)$ has the following properties:
  1. $c(0, \theta) = 0$ (no education implies not cost)
  2. $c_e(e, \theta) > 0$, $c_{ee}(e, \theta) > 0$
  3. $c_\theta(e, \theta) < 0 \quad \forall e > 0$
  4. $c_{e\theta}(e, \theta) < 0$
- w wage rate
- Utility of type θ: $u(w, e|\theta) \stackrel{\text{def}}{=} w - c(e, \theta)$
- Firms and workers are risk neutral
- $\mu(e) \in [0, 1]$ is firms' belief that the worker is of type θ^H
- Assumption: firms play Bertrand, so the wage drops to average productivity: $w(e) = \mu(e)\theta^H + [1 - \mu(e)]\theta^L$
- 3-stage game:
  1. Nature draws $\theta \in \{\theta^L, \theta^H\}$ (ignored)
  2. Workers decide on e (as function of θ)
  3. Firms derive belief function $\mu(e) \in [0, 1]$
  4. Firms offer wage (Bertrand)
- Looking for a Subgame-Perfect Equilibrium (called *perfect Bayesian equilibrium, PBE*)
- Two types of PBE: *separating* and *pooling*

Separating equilibrium

Lemma 5.1 *In a separating PBE, $\hat{w}(\hat{e}(\theta^H)) = \theta^H$ and $\hat{w}(\hat{e}(\theta^L)) = \theta^L$*

Proof. Beliefs must be correct in equilibrium, hence, wage drop to marginal product. ∎

Lemma 5.2 *In a separating PBE, $\hat{e}(\theta^L) = 0$ (i.e., no education).*

Proof. Anyway he gets $\hat{w}(\hat{e}(\theta^L)) = \theta^L$, so to minimize cost it must set $\hat{e}(\theta^L) = 0$. ∎

The equilibrium is: $\hat{e}(\theta^L) = 0$, $\hat{e}(\theta^H) = \tilde{e})$ where

$$\tilde{e} = \text{argmin}_e u(\theta^H, e|\theta^L) \leq u(\theta^L, 0|\theta^L)$$

And

$$\hat{\mu}(e) = \frac{\hat{w}(e) - \theta^L}{\theta^H - \theta^L}$$

Hence, $\hat{\mu}(0) = 0$ and $\hat{\mu}(\tilde{e}) = 1$.

Pooling equilibrium

For $\tilde{e}$ low enough

$$\hat{e}(\theta^L) = \hat{e}(\theta^H) = \tilde{e} \quad \hat{w}(\tilde{e}) = \lambda\theta^H + (1-\lambda)\theta^L$$

Low enough means to satisfy

$$u(\hat{w}, \tilde{e}|\theta^L) \geq 0$$

Remark: Pareto dominated by $\tilde{e} = 0$.

5.2 Adverse selection

5.2.1 An illustration of the Lemon Problem

- Akerlof (1970)
- four possible types of cars: brand-new good cars, brand-new lemon cars (bad cars), used good cars, and used lemon cars.
- All individuals in this economy have the same preferences:

  N^G = value of a new good car;
  N^L = value of a new lemon car;
  U^G = value of a used good car; and
  U^L = value of a used lemon car.

  We make the following assumptions:

Assumption 5.1

1. *The value of new and old lemon cars is zero; that is,* $N^L = U^L = 0$.
2. *Half of all cars (new and old) are lemons, and half are good cars.*
3. *New good cars are preferred over used good cars; that is,* $N^G > U^G > 0$.

Assumption 5.1 implies

$$\mathrm{E}N \equiv 0.5N^G + 0.5N^L = 0.5N^G > \mathrm{E}U \equiv 0.5U^G + 0.5U^L = 0.5U^G.$$

There are 4 types of agents in this economy:

1. new car dealers who sell new cars for an *exogenously given* uniform price denoted by p^N.
2. individuals who do not own any car, whom we call buyers in what follows;
3. owners of good used cars, whom we call sellers; and
4. owners of lemon used cars, whom we also call sellers.

Utility of a buyer (not an owner)

$$V^b \equiv \begin{cases} \mathrm{E}N - p^N & \text{if he buys a new car} \\ \mathrm{E}U - p^U & \text{if he buys a used car.} \end{cases}$$

The utility of a seller of a good used car who sells his used car for p^U and buys a new car for p^N is given by

$$V^{s,G} \equiv \begin{cases} \mathrm{E}N - p^N + p^U & \text{if he buys a new car (and sells his used car)} \\ U^G & \text{if he maintains his (good) used car.} \end{cases}$$

Finally, the utility of a seller of a lemon car who sells his used lemon for p^U and buys a new car for p^N is given by

$$V^{s,L} \equiv \begin{cases} \mathrm{E}N - p^N + p^U & \text{if he buys a new car (and sells his used car)} \\ U^L & \text{if he maintains his lemon used car.} \end{cases}$$

The problem of the buyers

Buy a used car if $\mathrm{E}U - p^U \geq \mathrm{E}N - p^N$, or if p^U satisfies

$$p^U \leq \mathrm{E}U - \mathrm{E}N + p^N = \frac{U^G - N^G}{2} + p^N.$$

The problem of the lemon used-car seller

Owner of a lemon used car sells his car if $0 \leq \mathrm{E}N - p^N + p^U$ or

$$p^U \geq p^N - \mathrm{E}N = p^N - 0.5N^G.$$

The problem of the good used-car seller

An owner of a good used car sells his car if $U^G \leq \mathrm{E}N - p^N + p^U$, or

$$p^U \geq p^N + U^G - EN = p^N + U^G - 0.5N^G. \tag{5.4}$$

Figure 5.2 summarizes the cases.

Figure 5.2 shows that the combinations of p^N and p^U satisfying the condition in which good used cars are sold do not satisfy the condition in which buyers would demand used cars.

Proposition 5.4 *Good used cars are never sold. That is, lemon used cars drive good used cars out of the market.*

5.2.2 Adverse selection in the labor market

- Market failure occurs because less productive workers are employed at a competitive wage
- industry with 1 product ($p = 1$)
- $\theta \in [L, H]$ is a worker's productivity

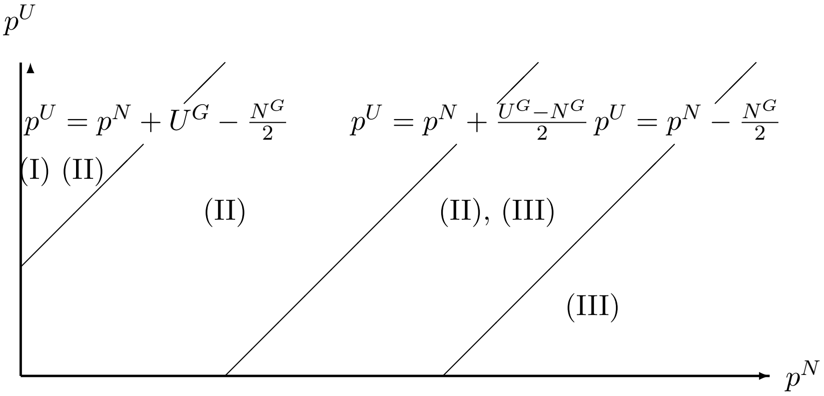

Figure 5.2: The market for lemons: Bad cars drive out the good cars. The prices of new and used cars corresponding to cases where used cars are demanded or offered for sale. (I) Good used-car seller sells. (II) Bad used-car seller sells. (III) Buyers demand used cars. (*Note:* The Figure assumes $U^G > N^G/2$).

- $F(\theta)$ is CDF (# workers with productivity $< \theta$
- Assume $F(H) = N$ (total # workers in the economy)
- $r(\theta)$ is reservation wage
- Worker type-θ accepts a wage-offer w if $r(\theta) \leq w$.
- Hence, the set of workers is $\Theta(w) \stackrel{\text{def}}{=} \{\theta | r(\theta) \leq w\}$
- Firms know distribution but not actual θ of a worker
- $\mu =$ firm's belief of average productivity
- Risk-neutral firm, under CRS the labor demand function is

$$q^d = \begin{cases} 0 & \text{if } \mu < w \\ & \text{if } \mu = w \\ \infty & \text{if } \mu > w \end{cases}$$

Definition 5.1 *The set of workers who accept employment $\hat{\Theta}$ and the wage rate $\hat{w}$ are called **competitive equilibrium with rational expectation** if*

$$\hat{\Theta} = \{\theta | r(\theta) \leq \hat{w}\} \quad \textit{and} \quad \hat{w} = E[\theta | \theta \in \hat{\Theta}]$$

Proposition 5.5 *An equilibrium need not be efficient*

Proof.

- For example, take $r(\theta) = r$ for all θ.

- If $E\theta > r$, then all workers accept employment, but, all $\theta < w = E\theta$ should not be employed.
- If $E\theta < r$, then no worker accepts employment but, all $\theta > w = E\theta$ should be employed.

■

Adverse selection is an additional potential market failure, happens when

ASSUMPTION 5.2 *Workers who are more productive at the firm are also more productive at home (or alternative employment place). Formally,*

$$\frac{dr(\theta)}{d\theta} > 0, \quad \forall \theta \in [L, H]$$

In equilibrium,

$$\hat{w} = E[\theta | r(\theta) \leq \hat{w}$$

implying that

Proposition 5.6 *The most productive workers do not accept employment*

Proof. In equilibrium, the set of those who accept employment is

$$\hat{\Theta} = \{\theta | r(\theta) \leq \hat{w}\}$$

Moreover, to get the most productive workers to accept the job offer, the wage rate needs to be $w = H$, yielding a loss. ■

5.3 The Principal-Agent Problem

- Due to lack of monitoring, some asymmetric information problem exist (illegality of certain monitoring techniques)
- Agents can write contracts taking into account of future asymmetric information
- So, this issue is about writing contracts
- Examples for principal-agent problems: Owner-manager; university-teachers; Owners-waiters
- Agent: waiter with effort level $e \in \{0, 2\}$

$$U = \begin{cases} Ew - e & \text{if he devotes an effort level } e \\ 10 & \text{if he works at another place.} \end{cases}$$

- Production: Restaurant's revenue is:

$$R(2) = \begin{cases} H & \text{probability } 0.8 \\ L & \text{probability } 0.2 \end{cases} \text{ and } R(0) = \begin{cases} H & \text{probability } 0.4 \\ L & \text{probability } 0.6. \end{cases}$$

- The Principal earns

$$\pi \equiv R(e) - w.$$

- The contract: on observed variable (revenue). Also depends on what level of effort the owner wants to implement.
- The *participation constraint* ($e = 2$ implementation):

$$0.8w^H + 0.2w^L - 2 \geq 10$$

- The *incentive constraint* ($e = 2$ implementation):

$$0.8w^H + 0.2w^L - 2 \geq 0.4w^H + 0.6w^L - 0$$

- The first equation implies that $w^L = 60 - 4w^H$, and the second implies that $w^L = w^H - 5$. Altogether, the optimal contract is $w^H = 13$ and $w^L = 8$.
- Hence, expected wage bill: $0.8 \times 13 + 0.2 \times 8 = 12$ (i.e., $10 + 2$ because of risk neutrality)
- Owners' profit under $e = 2$ implementation is:

$$ER - Ew = 0.8H + 0.2L - 0.8 \times 13 - 0.2 \times 8 = 0.8H + 0.2L - 12 \geq 0 \quad \text{if} \quad H > \frac{L}{4} - 15$$

- Who said that $e = 0$ implementation is most profitable?
- Set $w^H = w^L = 10$
- Participation constraint: Satisfied trivially
- Incentive constraint: Satisfied as $10 - 2 < 10 - 0$
- Profit under the $e = 0$ implementation $0.4H + 0.6L - 10$
- The $e = 2$ implementation yields a higher profit than the $e = 0$ implementation if

$$0.8H + 0.2L - 12 > 0.4H + 0.6L - 10 \quad \text{or} \quad H \geq L + 5$$

5.3.1 Risk-averse manager

- Let the waiter be risk averse. More precisely,

$$U = \begin{cases} E\sqrt{w - e} & \text{if he devotes an effort level } e \\ \sqrt{10} & \text{if he works at another place.} \end{cases}$$

- Participation constraint:

$$0.8\sqrt{w^H - 2} + 0.2\sqrt{w^L - 2} \geq \sqrt{10}$$

- Incentive constraint:

$$0.8\sqrt{w^H - 2} + 0.2\sqrt{w^L - 2} \geq 0.4\sqrt{w^H - 0} + 0.6\sqrt{w^L - 0}$$

- Risk averse waiter needs a higher expected compensation because substituting $w^H = 13$ and $w^L = 8$ into the participation constraint,

$$0.8\sqrt{13 - 2} + 0.2\sqrt{8 - 2} \approx 3.14 < \sqrt{10} \approx 3.16$$